

On Agaian Matrix

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In this paper we discuss the isolation problem of the Agaian matrix among the 6-dimensional matrices. In general a matrix isolation could depend on the kind of equivalence one uses for complex Hadamard matrices. However the present form of Agaian matrix has such a big symmetry that both equivalence methods lead to the same result: the matrix is isolated.

1. INTRODUCTION

One of the most interesting 6-dimensional matrices is the Agaian matrix which appears in [1] on page 112. By using the usual equivalence of complex Hadamard matrices this matrix is an isolated one. It has the form

$$A_1(\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega^2 & \omega^2 & \omega \\ 1 & \omega & 1 & \omega & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^2 & \omega & 1 & \omega \\ 1 & \omega & \omega^2 & \omega^2 & \omega & 1 \end{bmatrix} \quad (1)$$

where 1, ω and ω^2 are the solutions of equation $x^3 - 1 = 0$, and $i = \sqrt{-1}$.

Nowadays one makes use of two different equivalence methods: the standard method and the unitary equivalence. The first one has its origin in Sylvester paper [5] who introduced the so called standard form for real Hadamard matrices, which implies that the entries of the first row and column are equal to 1. This equivalence was also extended to complex Hadamard matrices. It says that two Hadamard matrices H_1 and H_2 are equivalent, written as $H_1 \equiv H_2$, if there exist two diagonal unitary matrices D_1 and D_2 , and permutation matrices P_1 and P_2 , such that

$$H_1 = D_1 P_1 H_2 P_2 D_2 \quad (2)$$

However the complex Hadamard matrices naturally belong to the class of normal matrices. A matrix N is normal if it commutes with its adjoint N^* , i.e. it satisfies the relation $N N^* = N^* N$. For this class of operators the unitary equivalence takes a simple form and says that every normal matrix is similar to a diagonal matrix D , which means that there exists a unitary matrix U , such that

$$N = U D U^* \quad (3)$$

see [4] p. 357. For all unitary matrices the entries of the diagonal matrix D are unimodular.

Two important classes of normal operators are the unitary and self-adjoint matrices, such that for both these classes two matrices M_1 and M_2 are unitary equivalent iff they have the same spectrum, or equivalently, the characteristic polynomials are the same up to a multiplicative constant factor, see [4], i.e.

$$pol(M_1(x)) = det(x I_n - M_1/\sqrt{n}) = det(x I_n - M_2/\sqrt{n}) = pol(M_2(x)) \quad (4)$$

where det is the determinant of the corresponding matrix. In the following we shall make use of this type of equivalence. To understand better the difference between the two equivalence methods we give a simple example of a selfadjoint Hadamard matrix

$$M_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & i & i & -i & -i \\ 1 & -i & -1 & 1 & -1 & i \\ 1 & -i & 1 & -1 & i & -1 \\ 1 & i & -1 & -i & 1 & -1 \\ 1 & i & -i & -1 & -1 & 1 \end{bmatrix} \quad (5)$$

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The unitary equivalence implies the knowledge of its spectrum, which in this case is given by

$$Sp(M_6) = [-1^3, 1^3] \quad (6)$$

where power means eigenvalue multiplicity. The above result shows that M_6 is a complex Hadamard matrix, and in the same time a selfadoint matrix, its entries and spectrum being unimodular and real.

If we make use of the standard equivalence (2) the following matrix

$$M_{61} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & i & -i \\ 1 & 1 & -1 & i & -1 & -i \\ 1 & -i & -1 & -1 & 1 & i \\ 1 & -1 & -i & 1 & -1 & i \\ 1 & i & i & -i & -i & -1 \end{bmatrix} \quad (7)$$

is equivalent to (5), but its spectrum is given by

$$Sp(M_{61}) = \left[-1^2, 1^2, \frac{i - \sqrt{2}}{\sqrt{3}}, -\frac{i + \sqrt{2}}{\sqrt{3}} \right] \quad (8)$$

Thus when one makes use of the usual equivalence, see relation (2), the classical quantum mechanics will be in a big danger, because the M_{61} eigenvalues are unimodular, but not real.

In the first case $M_6 = M_6^*$, and in the second case $M_{61} \neq M_{61}^*$. Thus the change of rows and/or columns between themselves modifies the matrix symmetry, and accordingly its spectrum.

2. AGAIAN MATRIX

The Agaian matrix, (1), is a Butson type matrix, see papers [2]-[3], and its first form appeared on page 104, in book [1], Chapter 2, paragraph 5, named Generalized Hadamard Matrices. Agaian notation is

$$H(3,6) = \begin{bmatrix} z & x & y & y & x & z \\ x & z & y & x & y & z \\ x & x & x & x & x & x \\ z & x & z & x & y & y \\ x & z & z & y & x & y \\ z & z & x & y & y & x \end{bmatrix} \quad (9)$$

where $x = 1$, $y = \omega$, $z = \omega^2$ are the third roots of unity, and it does not coincide with matrices from his note 5.1 where Agaian gave a few examples of generalized Hadamard matrices.

After providing the above matrix (9) Agaian said that each generalized Hadamard matrix can be brought to a standard form matrix, i.e. with first row and column entries equal to 1. After that he wrote:

Butson (1962) proved that for prime numbers p the necessary condition for existence of normalized $H(p,h)$ -matrices is $h = pt$ where t is a natural number.

In fact in both the papers, [2] and [3], are given only necessary conditions for the existence of a matrix whose entries come from Fourier matrix, and no numerical example is provided. Im our opinion this means that the above matrix was firstly obtained by Agaian.

With the above notations the $H(3,6)$ matrix (9) has the form

$$A_{10} = \begin{bmatrix} \omega^2 & 1 & \omega & \omega & 1 & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega & \omega^2 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \omega^2 & 1 & \omega^2 & 1 & \omega & \omega \\ 1 & \omega^2 & \omega^2 & \omega & 1 & \omega \\ \omega^2 & \omega^2 & 1 & \omega & \omega & 1 \end{bmatrix} \quad (10)$$

Because we make use of unitary equivalence we compute the spectral function of A_{10} -matrix and its form is given by

$$f(A_{10}) = x^6 + \sqrt{\frac{2}{3}}(\omega - 1)x^5 - \frac{1+2\omega}{2}x^4 + \frac{1+2\omega}{\sqrt{6}}x^3 - \frac{1+2\omega}{2}x^2 + \sqrt{\frac{2}{3}}(2+\omega)x - 1 \quad (11)$$

whose spectrum cannot be explicitly found because the above equation is not a reciprocal one. However its spectrum can be calculated numerically by using Mathematica, and its spectrum is simple.

Multiplying at left and right by unitary diagonal matrices generated by the first row and/or the first column the A_{10} matrix transforms into a standard form matrix

$$A_{01} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega & 1 & \omega^2 \\ 1 & \omega^2 & \omega & \omega & \omega^2 & 1 \\ 1 & 1 & \omega & \omega^2 & \omega & \omega^2 \\ 1 & \omega & 1 & \omega^2 & \omega^2 & \omega \\ 1 & \omega^2 & \omega^2 & 1 & \omega & \omega \end{bmatrix} \quad (12)$$

whose spectral function is simpler than (11) and has the form

$$f(A_{01}) = (x^2 - 1)(x^4 + \frac{1-\omega}{\sqrt{6}}x^3 + x^2 + \frac{\omega+2}{\sqrt{6}}x + 1) \quad (13)$$

In this case its spectrum can be easily found.

Perhaps the above form (12) was considered as being not sufficiently symmetric, and it was easily transformed into matrix (1) whose spectrum does not depend on ω , and is given by

$$Sp(A_1(\omega)) = \left[-1, 1, \left(\frac{\sqrt{3} - i\sqrt{5}}{2\sqrt{2}} \right)^2, \left(\frac{\sqrt{3} + i\sqrt{5}}{2\sqrt{2}} \right)^2 \right] \quad (14)$$

Looking at matrices (9) and (10) one observes that Again used only a particular case, namely that when $x = 1$, $y = \omega$, $z = \omega^2$. However the three parameters, (x, y, z) , generate six permutations and this suggests that there could be at least six different matrices, if one makes use of unitary equivalence, instead of the usual one.

For example if we take $x = \omega$, $y = 1$, $z = \omega^2$ in matrix (9) the form of the corresponding matrix is

$$A_{20} = \begin{bmatrix} \omega^2 & \omega & 1 & 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 & \omega & 1 & \omega^2 \\ \omega & \omega & \omega & \omega & \omega & \omega \\ \omega^2 & \omega & \omega^2 & \omega & 1 & 1 \\ \omega & \omega^2 & \omega^2 & 1 & \omega & 1 \\ \omega^2 & \omega^2 & \omega & 1 & 1 & \omega \end{bmatrix} \quad (15)$$

One sees that $A_{10} \neq A_{20}$. In fact the spectral function of (15) has the form

$$f(A_{20}) = x^6 + \sqrt{\frac{2}{3}}(1 - \omega)x^5 + \frac{1-\omega}{2}x^4 - \frac{1+2\omega}{\sqrt{6}}x^3 - \frac{\omega+2}{2}x^2 - \sqrt{\frac{2}{3}}(\omega+2)x - 1 \quad (16)$$

The corresponding standard form is

$$A_{02} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 & \omega^2 & \omega & 1 \\ 1 & 1 & \omega^2 & \omega & \omega^2 & \omega \\ 1 & \omega^2 & 1 & \omega & \omega & \omega^2 \\ 1 & \omega & \omega & 1 & \omega^2 & \omega^2 \end{bmatrix} \quad (17)$$

and its spectral function is given by

$$f(A_{02}) = (x^2 - 1)(x^4 + \frac{\omega + 2}{\sqrt{6}}x^3 + x^2 + \frac{1 - \omega}{\sqrt{6}}x + 1) \quad (18)$$

Another form of the matrix (17) is the following

$$A_2(\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega^2 & \omega & \omega^2 & \omega \\ 1 & \omega^2 & 1 & \omega & \omega & \omega^2 \\ 1 & \omega & \omega & 1 & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega & \omega & 1 & \omega^2 \\ 1 & \omega^2 & \omega & \omega^2 & \omega & 1 \end{bmatrix} \quad (19)$$

and it does not coincide with the A_1 -matrix, see (1); however its spectrum is given by the same relation (14).

For the choice $x = \omega$, $y = \omega^2$, $z = 1$ one finds

$$A_{30} = \begin{bmatrix} 1 & \omega & \omega^2 & \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 & \omega & \omega^2 & 1 \\ \omega & \omega & \omega & \omega & \omega & \omega \\ 1 & \omega & 1 & \omega & \omega^2 & \omega^2 \\ \omega & 1 & 1 & \omega^2 & \omega & \omega^2 \\ 1 & 1 & \omega & \omega^2 & \omega^2 & \omega \end{bmatrix} \quad (20)$$

and its spectral function is

$$f(A_{30}) = x^6 - \sqrt{\frac{2}{3}}(1 + 2\omega)x^5 + \frac{\omega - 1}{2}x^4 + \frac{1 + 2\omega}{\sqrt{6}}x^3 + \frac{\omega + 2}{2}x^2 - \sqrt{\frac{2}{3}}(1 + 2\omega)x - 1 \quad (21)$$

Similar to the preceding cases one gets

$$A_{03} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega & 1 & \omega^2 \\ 1 & \omega^2 & \omega & \omega & \omega^2 & 1 \\ 1 & 1 & \omega & \omega^2 & \omega & \omega^2 \\ 1 & \omega & 1 & \omega^2 & \omega^2 & \omega \\ 1 & \omega^2 & \omega^2 & 1 & \omega & \omega \end{bmatrix} \quad (22)$$

with the spectral function

$$f(A_{03}) = (x^2 - 1)(x^4 + \frac{1 - \omega}{\sqrt{6}}x^3 + x^2 + \frac{\omega + 2}{\sqrt{6}}x + 1) \quad (23)$$

From relations (13) and (23) we observe that $f(A_{01}) = f(A_{03})$, which means that from the six standard forms one gets only two different spectra given by equations (13) and (18). Thus at this level there are only two matrices that are not unitary equivalent.

The A_3 matrix has the form

$$A_3(\omega) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega^2 & \omega & \omega^2 \\ 1 & \omega & 1 & \omega^2 & \omega^2 & \omega \\ 1 & \omega^2 & \omega^2 & 1 & \omega & \omega \\ 1 & \omega & \omega^2 & \omega & 1 & \omega^2 \\ 1 & \omega^2 & \omega & \omega & \omega^2 & 1 \end{bmatrix} \quad (24)$$

The choice $x = \omega^2$, $y = \omega$, $z = 1$ leads to

$$A_{40} = \begin{bmatrix} 1 & \omega^2 & \omega & \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega & \omega^2 & \omega & 1 \\ \omega^2 & \omega^2 & \omega^2 & \omega^2 & \omega^2 & \omega^2 \\ 1 & \omega^2 & 1 & \omega^2 & \omega & \omega \\ \omega^2 & 1 & 1 & \omega & \omega^2 & \omega \\ 1 & 1 & \omega^2 & \omega^2 & \omega & \omega^2 \end{bmatrix} \quad (25)$$

whose spectral function is the following

$$f(A_{40}) = x^6 + \sqrt{\frac{2}{3}}(1+2\omega)x^5 - \frac{\omega+2}{2}x^4 - \frac{1+2\omega}{\sqrt{6}}x^3 + \frac{1-\omega}{2}x^2 + \sqrt{\frac{2}{3}}(1+2\omega)x - 1 \quad (26)$$

The choice $x = \omega^2$, $y = 1$, $z = \omega$ leads to A_{50} matrix whose form is

$$A_{50} = \begin{bmatrix} \omega & \omega^2 & 1 & 1 & \omega^2 & \omega \\ \omega^2 & \omega & 1 & \omega^2 & 1 & \omega \\ \omega^2 & \omega^2 & \omega^2 & \omega^2 & \omega^2 & \omega^2 \\ \omega & \omega^2 & \omega & \omega^2 & 1 & 1 \\ \omega^2 & \omega & \omega & 1 & \omega^2 & 1 \\ \omega & \omega & \omega^2 & 1 & 1 & \omega^2 \end{bmatrix} \quad (27)$$

and its spectral function is given by

$$f(A_{50}) = x^6 + \sqrt{\frac{2}{3}}(\omega+2)x^5 + \frac{\omega+2}{2}x^4 + \frac{2\omega+1}{\sqrt{6}}x^3 + \frac{\omega-1}{2}x^2 + \sqrt{\frac{2}{3}}(\omega-1)x - 1 \quad (28)$$

The parameters $x = 1$, $y = \omega^2$, $z = \omega$ give rise to the matrix

$$A_{60} = \begin{bmatrix} \omega & 1 & \omega^2 & \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 & 1 & \omega^2 & \omega \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \omega & 1 & \omega & 1 & \omega^2 & \omega^2 \\ 1 & \omega & \omega & \omega^2 & 1 & \omega^2 \\ \omega & \omega & 1 & \omega^2 & \omega^2 & 1 \end{bmatrix} \quad (29)$$

with the spectral function

$$f(A_{60}) = x^6 - \sqrt{\frac{2}{3}}(\omega+2)x^5 + \frac{2\omega+1}{2}x^4 - \frac{2\omega+1}{\sqrt{6}}x^3 + \frac{2\omega+1}{2}x^2 - \sqrt{\frac{2}{3}}(\omega-1)x - 1 \quad (30)$$

From the above examples one sees that $f(A_{i0}) \neq f(A_{j0})$, $i \neq j$, $i, j = 1, \dots, 6$.

On the other hand all the six matrices A_i are different since $A_i \neq A_j$, $i \neq j$, (we have written only three of them), have the same spectrum given by relation (14), thus they are *unitary equivalent*, and the matrix (1) *is indeed an isolated one*. Why happens so? An explanation could be the following: the above matrices A_i can be transformed into symmetric matrices by the choice $\omega = a$, where a is a real number. For example A_2 has the new symmetric form

$$A_2(a) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & a^2 & a & a^2 & a \\ 1 & a^2 & 1 & a & a & a^2 \\ 1 & a & a & 1 & a^2 & a^2 \\ 1 & a^2 & a & a & 1 & a^2 \\ 1 & a^2 & a & a^2 & a & 1 \end{bmatrix} \quad (31)$$

and its spectrum is given by

$$Sp(A_2(a)) = \left[\frac{1 + a + a^2 \pm \sqrt{a^2(1 + a^2) + 5}}{\sqrt{6}}, \left(\frac{\pm \sqrt{5a^2(a - 1)^2 + 2 - a(1 + a)}}{2\sqrt{6}} \right)^2 \right] \quad (32)$$

and all the $A_i(a)$ matrices lead to the same spectrum, even if the matrices do not coincide. This thing could be interpreted as a higher symmetry of Agaian matrix since now a is an arbitrary real number.

3. CONCLUSION

All the six forms of $A_{i0}(\omega)$ matrices, $i = 1, \dots, 6$, obtained from matrix (9) are *not unitary equivalent*. When they are brought to their standard form only two of them are not unitary equivalent.

The multiplication of $A_{i0}(\omega)$ matrices by diagonal matrices generated by their first row and column leads to a higher symmetry.

Thus from the unitary equivalence point of view we have *six* Agaian matrices of the form (10), (15), etc, which are not unitary equivalent, and only *two* Agaian matrices of the form (12), (17), which are brought to their standard form and which are not unitary equivalent between themselves.

The final conclusion is that all the six form of the Agaian matrix whose first diagonal entries are equal to 1 are equivalent, irrespective of the equivalence method which is used for. This shows that the Agaian matrix is isolated.

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